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Revisiting the $Y = 0$ open spin chain at one loop

Rafael I. Nepomechie [1]

Physics Department, P.O. Box 248046, University of Miami

Coral Gables, FL 33124 USA

Abstract

In 2005, Berenstein and Vázquez determined an open spin chain Hamiltonian describing the one-loop anomalous dimensions of determinant-like operators corresponding to open strings attached to $Y = 0$ maximal giant gravitons. We construct the transfer matrix (generating functional of conserved quantities) containing this Hamiltonian, thereby directly proving its integrability. We find the eigenvalues of this transfer matrix and the corresponding Bethe equations, which we compare with proposed all-loop Bethe equations. We note that the Bethe ansatz solution has a certain "gauge" freedom, and is not completely unique.

[1] nepomechie@physics.miami.edu

1 Introduction

The discovery and exploitation of integrability in planar AdS/CFT has already led to many remarkable results [1], and may even ultimately lead to the solution of planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM), which is widely regarded as the "harmonic oscillator" of 4-dimensional gauge theories. Much of the focus has been on the problem of computing the anomalous dimensions of single-trace operators of $\mathcal{N} = 4$ SYM, which can be mapped to the problem of determining the eigenvalues of certain integrable *closed* spin-chain Hamiltonians, as observed in the seminal work of Minahan and Zarembo [2]. [1] However, progress has also been made on the problem of computing the anomalous dimensions of determinant-like operators, which can be mapped to the problem of determining the eigenvalues of certain *open* spin-chain Hamiltonians [6]. By the AdS/CFT correspondence, these two types of operators correspond to states of closed strings and open strings attached to D-branes, respectively.

The simplest and most-studied open string/chain example is the so-called $Y = 0$ maximal giant graviton brane [7]. The one-loop $SO(6)$ scalar sector open spin chain Hamiltonian was found by Berenstein and Vázquez [8]. They also determined the (one-loop) boundary S-matrix, and showed that it satisfies the boundary Yang-Baxter equation (BYBE) [9, 10, 11]. A breakdown of integrability in the $SU(2)$ subsector at two loops was suspected in [12, 13]. However, Hofman and Maldacena [7] subsequently showed that integrability is in fact preserved at two loops; and, based on $SU(1|2)$ symmetry, they proposed an all-loop boundary S-matrix that satisfies the BYBE. (See also [14].) The scalar factor of the all-loop boundary S-matrix was proposed in [15]. Corresponding all-loop Bethe equations were proposed by Galleas in [16]. The bulk worldsheet Yangian symmetry discovered in [17], suitably generalized to the case of boundary scattering [18, 19], was used to construct all-loop bound-state boundary S-matrices [18, 20]. Finite-size corrections have been considered in [21, 22]. The classical integrability of the corresponding string sigma model with boundary has been investigated in [23, 24].

Despite the availability of all-loop results, there are still some important unresolved problems at one loop (where results are generally most solid): the transfer matrix (generating functional of conserved quantities) containing the Berenstein-Vázquez Hamiltonian has not been constructed, and the corresponding eigenvalues and Bethe equations have not been determined. For the corresponding one-loop $SO(6)$ scalar sector closed spin chain, such results were already obtained in the original Minahan-Zarembo work [2].

The purpose of our paper is to fill this gap: namely, to construct the one-loop open-chain transfer matrix, and to determine its eigenvalues. In this way, we directly prove the integrability of the one-loop Hamiltonian [8], and test the all-loop Bethe equations [16]. The simpler case of the $SU(3)$ subsector was treated in [25, 26].

The outline of this paper is as follows. In Section 2, we recall the open spin chain Hamiltonian found in [8], and construct the corresponding transfer matrix. In Section 3 we use the analytical Bethe ansatz to determine the eigenvalues of this transfer matrix and the corresponding Bethe equations. Surprisingly, the Bethe ansatz solution has a certain

[1] We have in mind here "long" operators. For operators of finite length, there are finite-size corrections [3, 4, 5].

"gauge" freedom, and is not completely unique. One of the sets of one-loop Bethe equations that we find is consistent with the all-loop equations. We conclude in Section 4 with a brief discussion of our results.

2 Construction of the transfer matrix

In [8] Berenstein and Vázquez identified an open $SO(6)$ spin chain that describes the one-loop anomalous dimensions of determinant-like operators corresponding to open strings attached to $Y = 0$ maximal giant gravitons. The space of states is [2]

$$\overset{\overset{0}{\downarrow}}{C^5} \otimes \overset{\overset{1}{\downarrow}}{C^6} \otimes \cdots \overset{\overset{L}{\downarrow}}{C^6} \otimes \overset{\overset{L+1}{\downarrow}}{C^5} \; . \tag{1}$$

That is, the vector spaces of the "bulk" sites (labeled $1, \ldots, L$) all have dimension 6, while the vector spaces of the left and right "boundary" sites (labeled 0 and $L+1$, respectively) have dimension 5. The Hamiltonian is given in Eq. (2.15), which we rewrite as

$$H = Q_0^Y h_{0,1} Q_0^Y + (I - Q_0^{\bar{Y}}) + \sum_{l=1}^{L-1} h_{l,l+1} + Q_{L+1}^Y h_{L,L+1} Q_{L+1}^Y + (I - Q_{L+1}^{\bar{Y}}) \,, \tag{2}$$

where $h_{l,l+1}$ is the bulk two-site Hamiltonian

$$h_{l,l+1} = \frac{1}{2}\mathcal{K}_{l,l+1} + I_{l,l+1} - \mathcal{P}_{l,l+1} \,. \tag{3}$$

We have relabeled the sites to run from 0 to $L+1$ (instead of 1 to L); we have relabeled Z and $\bar{Z}$ by Y and $\bar{Y}$, respectively; and for simplicity, we have set the coupling constant $\lambda \equiv 1$. We note that Q^ϕ is the projector

$$Q^\phi|\phi\rangle = 0 \,, \quad Q^\phi|\varphi\rangle = |\varphi\rangle \text{ for } \varphi \neq \phi \,. \tag{4}$$

In the standard basis $|a\rangle = e_a$ with $a = 1, \ldots, 6$ (elementary 6-dimensional vectors with components $\left[e_a\right]_i = \delta_{a,i}$), the matrices $\mathcal{P}$ and $\mathcal{K}$ are given by

$$\mathcal{P} = \sum_{a,b=1}^{6} e_{ab} \otimes e_{ba} \,, \qquad \mathcal{K} = \sum_{a,b=1}^{6} e_{ab} \otimes e_{ab} \,, \tag{5}$$

where e_{ab} is the standard elementary 6×6 matrix whose (a,b) matrix element is 1, and all others are zero; i.e., $\left[e_{ab}\right]_{ij} = \delta_{ai}\delta_{bj}$.

It is convenient to change to a new basis,

$$\begin{aligned}
|\tilde{1}\rangle &= |W\rangle = \frac{1}{\sqrt{2}}(e_1 + ie_2) \,, \qquad |\tilde{2}\rangle = |\bar{W}\rangle = \frac{1}{\sqrt{2}}(e_1 - ie_2) \,, \\
|\tilde{3}\rangle &= |Z\rangle = \frac{1}{\sqrt{2}}(e_3 + ie_4) \,, \qquad |\tilde{4}\rangle = |\bar{Z}\rangle = \frac{1}{\sqrt{2}}(e_3 - ie_4) \,, \\
|\tilde{5}\rangle &= |Y\rangle = \frac{1}{\sqrt{2}}(e_5 + ie_6) \,, \qquad |\tilde{6}\rangle = |\bar{Y}\rangle = \frac{1}{\sqrt{2}}(e_5 - ie_6) \,.
\end{aligned} \tag{6}$$

[2]Following [7], we define the origin of the spin chain at site 0 instead of site 1.

Let U be the unitary operator which implements this change of basis,

$$|\tilde{a}\rangle = U|a\rangle \,. \tag{7}$$

Its matrix elements in the original basis $U_{ba} = \langle b|U|a\rangle = \langle b|\tilde{a}\rangle$ are given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & & & & \\ i & -i & & & & \\ & & 1 & 1 & & \\ & & i & -i & & \\ & & & & 1 & 1 \\ & & & & i & -i \end{pmatrix} , \tag{8}$$

where matrix elements that are zero are left empty.

In this new basis, $\mathcal{P}$ does not change, but $\mathcal{K}$ does change:

$$\begin{aligned} \tilde{\mathcal{P}} &= (U^\dagger \otimes U^\dagger)\mathcal{P}(U \otimes U) = \mathcal{P} \,, \\ \tilde{\mathcal{K}} &= (U^\dagger \otimes U^\dagger)\mathcal{K}(U \otimes U) \,. \end{aligned} \tag{9}$$

Moreover, in this new basis, the projectors Q^Y and $Q^{\bar{Y}}$ are given by the diagonal matrices

$$\begin{aligned} Q^Y &= \text{diag}(1,1,1,1,0,1) = 1 - |Y\rangle\langle Y| \,, \\ Q^{\bar{Y}} &= \text{diag}(1,1,1,1,1,0) = 1 - |\bar{Y}\rangle\langle\bar{Y}| \,. \end{aligned} \tag{10}$$

We therefore arrive at the following explicit matrix representation of the Hamiltonian

$$H = H_{bt}^L + \sum_{l=1}^{L-1} \tilde{h}_{l,l+1} + H_{bt}^R \,, \tag{11}$$

where the bulk two-site Hamiltonian is given by

$$\tilde{h}_{l,l+1} = \frac{1}{2}\tilde{\mathcal{K}}_{l,l+1} + I_{l,l+1} - \mathcal{P}_{l,l+1} \,, \tag{12}$$

and the boundary terms are given by

$$H_{bt}^L = Q_0^Y \tilde{h}_{0,1} Q_0^Y + (I - Q_0^{\bar{Y}}) \,, \tag{13}$$

$$H_{bt}^R = Q_{L+1}^Y \tilde{h}_{L,L+1} Q_{L+1}^Y + (I - Q_{L+1}^{\bar{Y}}) \,. \tag{14}$$

The boundary terms have the property

$$H_{bt}^L = Q_0^Y \, H_{bt}^L \, Q_0^Y \,, \qquad H_{bt}^R = Q_{L+1}^Y \, H_{bt}^R \, Q_{L+1}^Y \,. \tag{15}$$

We drop the null rows and columns in these matrices that are due to the Q^Y projectors. Hence, H_{bt}^L and H_{bt}^R should be understood as 30×30 matrices acting on $C^5 \times C^6$ and $C^6 \times C^5$, respectively, as indicated in (1).

We now address the problem of constructing the corresponding transfer matrix that contains this Hamiltonian as well as the higher conserved charges. According to Sklyanin [10], in order to construct an open-chain transfer matrix, we need an R-matrix that gives the bulk two-site Hamiltonian; and also left and right K-matrices, which give the left and right boundary terms, respectively. [3]

2.1 $R(u)$ matrix

We recall that the Yang-Baxter equation is given by

$$R_{12}(u_1 - u_2)\, R_{13}(u_1)\, R_{23}(u_2) = R_{23}(u_2)\, R_{13}(u_1)\, R_{12}(u_1 - u_2)\,. \tag{16}$$

An $SO(6)$-invariant solution which acts on $C^6 \otimes C^6$ is given (in the new basis) by [27, 2]

$$R(u) = \frac{1}{n-2}\left[u(2u+2-n)I - (2u+2-n)\mathcal{P} + 2u\tilde{\mathcal{K}}\right]\,, \tag{17}$$

with $n = 6$. This R-matrix indeed produces the bulk two-site Hamiltonian (12), since

$$\tilde{h}_{l,l+1} = \mathcal{P}_{l,l+1} R'_{l,l+1}(0) + \frac{3}{2} I_{l,l+1}\,, \tag{18}$$

where the prime denotes differentiation with respect to the spectral parameter u.

2.2 $K^-(u)$ matrix

The right K-matrix should give the right boundary term (14) in the Hamiltonian. The K-matrix must therefore be operator-valued, rather than c-number valued. Indeed, the K-matrix $K^-(u)$ (which acts on $C^6 \otimes C^5$) must satisfy the right "operator" BYBE

$$\begin{aligned} &R_{12}(u_1 - u_2)\, K^-_{13}(u_1)\, R_{12}(u_1 + u_2)\, K^-_{23}(u_2) \\ &\quad = K^-_{23}(u_2)\, R_{12}(u_1 + u_2)\, K^-_{13}(u_1)\, R_{12}(u_1 - u_2)\,, \end{aligned} \tag{19}$$

where the R-matrix is given by (17).

We claim that the needed K-matrix is given by

$$K^-_{12}(u) = Q^Y_2 R_{12}(u) \tilde{K}_1(u) R_{12}(u) Q^Y_2\,, \tag{20}$$

where $\tilde{K}(u)$ is the 6×6 diagonal K-matrix [28, 29]

$$\tilde{K}(u) = \mathrm{diag}(1-u, 1-u, 1-u, 1-u, u-1, u+1)\,, \tag{21}$$

[3] In order to avoid confusion, it may be worth noting that these R and K matrices, even though they satisfy bulk and boundary Yang-Baxter equations, have no direct relation to the bulk and boundary S-matrices discussed in the Introduction. This fact is evident in the more familiar case of the ferromagnetic spin-1/2 XXX Heisenberg chain: while the bulk S-matrix is a $U(1)$ phase, the R-matrix is an $SU(2)$-invariant 4×4 matrix.

which satisfies the standard (not operator) BYBE

$$\begin{aligned} & R_{12}(u_1-u_2)\,\tilde{K}_1(u_1)\,R_{12}(u_1+u_2)\,\tilde{K}_2(u_2) \\ & \quad = \tilde{K}_2(u_2)\,R_{12}(u_1+u_2)\,\tilde{K}_1(u_1)\,R_{12}(u_1-u_2)\,. \end{aligned} \tag{22}$$

We note that operator K-matrices of the general type (20) were introduced by Frahm and Slavnov [30], who called them "projected" K-matrices. The fact that K-matrices of this type are needed to construct the $SU(3)$ subsector of the $Y=0$ spin chain was noted in [26].

Indeed, we have verified that (20) does satisfy the operator BYBE (19), and produces the right boundary term (14),

$$H^R_{bt} = \frac{1}{2} K^{-\,\prime}(0) + 2I\,. \tag{23}$$

We also note that

$$K^-(0) = I\,. \tag{24}$$

2.3 $K^+(u)$ matrix

The left K-matrix should give the left boundary term (13) in the Hamiltonian. For the $SU(3)$ case [25] and in fact for the general $SU(N)$ case, we found in [26] that the needed left K-matrix can be obtained from the right K-matrix using Sklyanin's [10] "less obvious" isomorphism [4]

$$K^+_{13}(u) = k(u)\,\mathrm{tr}_2\,\mathcal{P}_{12} R_{12}(-2u-\eta) K^-_{23}(u)\,, \tag{25}$$

where η appears in the crossing-unitarity relation

$$R_{12}(u)^{t_1}\,R_{12}(-u-\eta)^{t_1} \propto I\,, \tag{26}$$

and $k(u)$ is an arbitrary scalar factor.

Fortunately, the same trick works also for the $SO(6)$ case. Indeed, the $SO(6)$ R-matrix (17) satisfies the crossing-unitarity property (26) with $\eta=-4$. We have verified that the K-matrix $K^+(u)$ given by the isomorphism (25), which acts on $C^6 \otimes C^5$, satisfies the left operator BYBE

$$\begin{aligned} & R_{12}(-u_1+u_2)\,K^+_{13}(u_1)^{t_1}\,R_{12}(-u_1-u_2-\eta)\,K^+_{23}(u_2)^{t_2} \\ & \quad = K^+_{23}(u_2)^{t_2}\,R_{12}(-u_1-u_2-\eta)\,K^+_{13}(u_1)^{t_1}\,R_{12}(-u_1+u_2)\,. \end{aligned} \tag{27}$$

And, most importantly, this K-matrix produces the left boundary term (13),

$$H^L_{bt} = -\frac{1}{6}\left[\mathrm{tr}_a\,K^+_{a0}(0)\,(R'_{a1}(0)\mathcal{P}_{a1} + \mathcal{P}_{a1}R'_{a1}(0)) + \mathrm{tr}_a\,K^{+\,\prime}_{a0}(0)\right] + \frac{7}{3}I\,, \tag{28}$$

[4] While in [25] the left K-matrix is defined to act on V_0V_1, here we define the left K-matrix to act on V_1V_0; i.e., the two K-matrices are related simply by permutation of the vector spaces V_0 and V_1. The isomorphism (25) evidently gives directly the latter form.

where the trace is over a 6-dimensional auxiliary space, which is discussed further below. We have fixed the scalar factor in (25) to be $k(u) = [u(u-2)(2u-1)]^{-1}$ in order to cancel corresponding terms that appear in the numerator. Finally, we note that

$$\mathrm{tr}_a\, K^{+}_{a0}(0) = -3I\,. \tag{29}$$

2.4 Transfer matrix

Having gathered all the necessary ingredients, we are now ready to assemble them to form the transfer matrix. We introduce a 6-dimensional auxiliary, denoted by a, and consider operators on the enlarged vector space (cf. (1))

$$\overset{a}{\overset{\downarrow}{C^6}} \otimes \overset{0}{\overset{\downarrow}{C^5}} \otimes \overset{1}{\overset{\downarrow}{C^6}} \otimes \cdots \overset{L}{\overset{\downarrow}{C^6}} \otimes \overset{L+1}{\overset{\downarrow}{C^5}}\ . \tag{30}$$

We define the monodromy matrices T and $\hat{T}$ by

$$T_{a1\cdots L}(u) = R_{a1}(u)\cdots R_{aL}(u)\,, \qquad \hat{T}_{a1\cdots L}(u) = R_{aL}(u)\cdots R_{a1}(u)\,, \tag{31}$$

where $R(u)$ is given by (17). The transfer matrix is given by [10]

$$t(u) = \mathrm{tr}_a\, K^{+}_{a0}(u)\, T_{a1\cdots L}(u)\, K^{-}_{aL+1}(u)\, \hat{T}_{a1\cdots L}(u)\,, \tag{32}$$

where $K^-(u)$ and $K^+(u)$ are given by (20) and (25), respectively.

By construction, the transfer matrix has the fundamental commutativity property

$$[t(u)\,, t(v)] = 0\,, \tag{33}$$

and contains the Hamiltonian (11),

$$H = -\frac{1}{6}t'(0) + \left(\frac{3}{2}L + \frac{17}{6}\right) I\,. \tag{34}$$

The relations (33) and (34) directly imply the integrability of the Hamiltonian. Higher conserved quantities can be obtained from higher derivatives of the transfer matrix at $u = 0$.

We note that the transfer matrix $t(u)$ is crossing invariant up to a scalar factor,

$$t(2-u) = \frac{u}{2-u}t(u)\,. \tag{35}$$

Equivalently, defining a rescaled transfer matrix $\bar{t}(u)$ by

$$\bar{t}(u) = u\, t(u)\,, \tag{36}$$

we see that this rescaled transfer matrix is exactly crossing invariant,

$$\bar{t}(2-u) = \bar{t}(u)\,. \tag{37}$$

The Hamiltonian is evidently related to the *second* derivative of $\bar{t}(u)$ at $u = 0$.

3 Analytical Bethe ansatz

The commutativity property (33) implies that it is possible to find eigenstates $|\Lambda\rangle$ of the transfer matrix $t(u)$ (32) which are independent of u,

$$t(u)\,|\Lambda\rangle = \Lambda(u)\,|\Lambda\rangle\,. \tag{38}$$

We turn now to the problem of determining the eigenvalues $\Lambda(u)$. We proceed by the analytical Bethe ansatz approach, along the lines in [2, 25]. We choose as a reference state

$$|0\rangle = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}^{\otimes L} \otimes \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \tag{39}$$

which is a ground state of the Hamiltonian (11) with eigenvalue 0. We denote the corresponding eigenvalue of the transfer matrix by $\Lambda_0(u)$,

$$t(u)\,|0\rangle = \Lambda_0(u)\,|0\rangle\,. \tag{40}$$

On the basis of results for $L = 0, 1, 2$, we obtain the following conjecture for the vacuum eigenvalue

$$\Lambda_0(u) = \frac{1}{4^L d(u)}\Big[a(u)(u-2)^{2L}(u-1)^{2L} + b(u)(u-1)^{2L}u^{2L} + 4c(u)(u-2)^{2L}u^{2L}\Big]\,, \tag{41}$$

where

$$\begin{aligned} a(u) &= (u-2)^4(u-1)^5(2u-3)^2\,, \\ b(u) &= (u-2)(u-1)^5u^3(2u-1)^2\,, \\ c(u) &= 4(u-2)^4(u-1)^4u^3\,, \\ d(u) &= 16(2u-3)(2u-1)\,. \end{aligned} \tag{42}$$

A general eigenvalue should be a "dressed" vacuum eigenvalue, [5]

$$\begin{aligned} \Lambda(u) &= \frac{1}{4^L d(u)}\Big\{a(u)(u-2)^{2L}(u-1)^{2L}\frac{Q_1(u+\frac{1}{2})}{Q_1(u-\frac{1}{2})} + b(u)(u-1)^{2L}u^{2L}\frac{Q_1(u-\frac{5}{2})}{Q_1(u-\frac{3}{2})} \\ &+ (u-2)^{2L}u^{2L}\Big[c_1(u)\frac{Q_1(u-\frac{3}{2})Q_2(u)Q_3(u)}{Q_1(u-\frac{1}{2})Q_2(u-1)Q_3(u-1)} \\ &+ c_2(u)\frac{Q_1(u-\frac{1}{2})Q_2(u-2)Q_3(u-2)}{Q_1(u-\frac{3}{2})Q_2(u-1)Q_3(u-1)} \\ &+ c_3(u)\frac{Q_2(u)Q_3(u-2)}{Q_2(u-1)Q_3(u-1)} + c_4(u)\frac{Q_2(u-2)Q_3(u)}{Q_2(u-1)Q_3(u-1)}\Big]\Big\}\,, \end{aligned} \tag{43}$$

[5] For the corresponding closed-chain result, see (4.28) in [2] and references therein.

where

$$Q_a(u) \equiv \prod_{k=1}^{m_a} (u - iu_{a,k})(u + iu_{a,k}) \,, \quad a = 1, 2, 3, \tag{44}$$

which have the property $Q_a(-u) = Q_a(u)$. The functions $c_1(u), \ldots, c_4(u)$ must satisfy

$$c_1(u) + c_2(u) + c_3(u) + c_4(u) = 4c(u) \,, \tag{45}$$

but are otherwise still to be determined.

The eigenvalues of the Hamiltonian (11) now follow from (34) and (43),

$$E = -\frac{1}{6}\Lambda'(0) + \frac{3}{2}L + \frac{17}{6} = \sum_{k=1}^{m_1} \frac{1}{u_{1,k}^2 + \frac{1}{4}} \,, \tag{46}$$

which is the same expression that one obtains for the corresponding closed chain [2].

The crossing symmetry (35) implies a corresponding property of the eigenvalues

$$\Lambda(2-u) = \frac{u}{2-u}\Lambda(u) \,, \tag{47}$$

which in turn implies the constraints

$$c_1(2-u) = \frac{u}{2-u} c_2(u) \,, \qquad c_3(2-u) = \frac{u}{2-u} c_4(u) \,. \tag{48}$$

The Bethe equations for the zeros $u_{1,k}$ of the function $Q_1(u)$ (44) follow from the fact that $\Lambda(u)$ in (43) is analytic at $u = iu_{1,k} + \frac{1}{2}$, which implies

$$\left(\frac{iu_{1,k} + \frac{1}{2}}{iu_{1,k} - \frac{1}{2}}\right)^{2L} = -\frac{a(iu_{1,k} + \frac{1}{2})}{c_1(iu_{1,k} + \frac{1}{2})} \frac{Q_1(iu_{1,k} + 1)\, Q_2(iu_{1,k} - \frac{1}{2})\, Q_3(iu_{1,k} - \frac{1}{2})}{Q_1(iu_{1,k} - 1)\, Q_2(iu_{1,k} + \frac{1}{2})\, Q_3(iu_{1,k} + \frac{1}{2})} \,. \tag{49}$$

We should obtain the same Bethe equations by considering instead the poles at $u = iu_{1,k} + \frac{3}{2}$, which implies the constraint

$$\frac{a(u)}{c_1(u)} = \frac{c_2(u+1)}{b(u+1)} \,. \tag{50}$$

Similarly, by considering the poles at $u = iu_{2,k} + 1$, we obtain the Bethe equations

$$1 = -\frac{c_1(iu_{2,k} + 1)}{c_4(iu_{2,k} + 1)} \frac{Q_2(iu_{2,k} + 1)\, Q_1(iu_{2,k} - \frac{1}{2})}{Q_2(iu_{2,k} - 1)\, Q_1(iu_{2,k} + \frac{1}{2})} \,, \tag{51}$$

and the constraint

$$\frac{c_1(u)}{c_4(u)} = \frac{c_3(u)}{c_2(u)} \,. \tag{52}$$

Finally, by considering the poles at $u = iu_{3,k} + 1$, we obtain the Bethe equations

$$1 = -\frac{c_1(iu_{3,k}+1)}{c_3(iu_{3,k}+1)} \frac{Q_3(iu_{3,k}+1)\, Q_1(iu_{3,k}-\frac{1}{2})}{Q_3(iu_{3,k}-1)\, Q_1(iu_{3,k}+\frac{1}{2})}\,, \tag{53}$$

and the same constraint (52). It can be shown that the requirement that the eigenvalues (43) be analytic at $u = \frac{1}{2}, \frac{3}{2}$ (where $d(u)$ has zeros) does not lead to further constraints. [6]

The constraints (45), (48), (50), (52) do not uniquely determine the functions $c_1(u), \ldots, c_4(u)$. Moreover, the problem of determining the transfer-matrix eigenvalues $\Lambda(u)$ in the $T-Q$ form (43) does not have a unique solution. Indeed, consider the following ansatz for these functions

$$\begin{aligned} c_1(u) &= u^n (u-1)^{5-n} (u-2)^4 (2u-3)^2\,, \\ c_2(u) &= u^3 (u-1)^{5-n} (u-2)^{1+n} (2u-1)^2\,, \\ c_3(u) &= u^{3+m} (u-1)^{5-n} (u-2)^{1+n-m} (2u-1)(2u-3)\,, \\ c_4(u) &= u^{n-m} (u-1)^{5-n} (u-2)^{4+m} (2u-1)(2u-3)\,, \end{aligned} \tag{54}$$

where n and m are integers. All the constraints (45), (48), (50), (52) are then satisfied for the following four sets of (n, m) values

$$\begin{aligned} &\text{case I} &:& \quad n = 3\,, \quad m = 0\,, \\ &\text{case IIa} &:& \quad n = 5\,, \quad m = 0\,, \\ &\text{case IIb} &:& \quad n = 5\,, \quad m = 2\,, \\ &\text{case III} &:& \quad n = 7\,, \quad m = 2\,, \end{aligned} \tag{55}$$

which we have designated as cases I, IIa, IIb, III, respectively.

The Bethe equations (49), (51), (53) can now be rewritten in the more familiar form

$$\begin{aligned} e_1(u_{1,k})^{2L+n-1} &= \prod_{\substack{j=1\\ j\neq k}}^{m_1} e_2(u_{1,k}-u_{1,j})\, e_2(u_{1,k}+u_{1,j}) \prod_{j=1}^{m_2} e_{-1}(u_{1,k}-u_{2,j})\, e_{-1}(u_{1,k}+u_{2,j}) \\ &\times \prod_{j=1}^{m_3} e_{-1}(u_{1,k}-u_{3,j})\, e_{-1}(u_{1,k}+u_{3,j})\,, \quad k = 1\,, \ldots\,, m_1\,, \end{aligned} \tag{56}$$

$$\begin{aligned} 1 &= e_2(u_{2,k})^m \prod_{\substack{j=1\\ j\neq k}}^{m_2} e_2(u_{2,k}-u_{2,j})\, e_2(u_{2,k}+u_{2,j}) \\ &\times \prod_{j=1}^{m_1} e_{-1}(u_{2,k}-u_{1,j})\, e_{-1}(u_{2,k}+u_{1,j})\,, \quad k = 1\,, \ldots\,, m_2\,, \end{aligned} \tag{57}$$

[6] It may be possible to derive further constraints from the requirement that the eigenvalues of the transfer matrix obtained by fusion in the auxiliary space also be analytic. However, such calculations would be difficult, and we shall not pursue them here.

$$
\begin{aligned}
1 &= e_2(u_{3,k})^{n-m-3} \prod_{\substack{j=1\\ j\neq k}}^{m_3} e_2(u_{3,k}-u_{3,j})\, e_2(u_{3,k}+u_{3,j}) \\
&\quad \times \prod_{j=1}^{m_1} e_{-1}(u_{3,k}-u_{1,j})\, e_{-1}(u_{3,k}+u_{1,j})\,, \quad k=1\,,\ldots\,,m_3\,,
\end{aligned} \tag{58}
$$

where

$$
e_n(u) = \frac{u+\frac{in}{2}}{u-\frac{in}{2}}\,. \tag{59}
$$

For cases I and III, the Bethe equations are symmetric under the interchange of Bethe roots of types 2 and 3 (i.e., $u_{2,i} \leftrightarrow u_{3,i}$), as in the closed-chain case [2]. The Bethe equations for cases IIa and IIb transform into each other under the interchange of Bethe roots of types 2 and 3.

At least for small values of L, the Bethe ansatz solutions corresponding to each of the cases (55) are complete; i.e., they reproduce all the transfer-matrix eigenvalues, of which there are $6^L 5^2$ according to (1). Indeed, following the approach described in Appendix B of [25], we have verified the completeness for $L=0$ and $L=1$, for which the total number of states is 25 and 150, respectively. The results for case I are summarized in Table 1 and Table 2, respectively. For cases II and III, the Bethe roots of type 1 (i.e., $\{u_{1,k}\}$) describing a given eigenvalue are the same as for case I, but the Bethe roots of types 2 and 3 are different.[7] This does not lead to differences in the energy (46) or higher conserved quantities, which depend only on the Bethe roots of type 1. Hence, the various cases can be regarded as equivalent. Gauge-like transformations relating these cases are discussed in Section A.

For case I, the above one-loop Bethe equations coincide with those that we previously conjectured, and obtained from the all-loop Bethe equations [16] by performing the weak-coupling limit and then reducing to the $SO(6)$ sector, which are given in Eqs. (3.1) and (4.20) in [25], respectively. For cases II and III, the Bethe equations are evidently slightly different: in (56) the term $e_1(u_{1,k})$ has a different power; and (57) and/or (58) contain an additional factor e_2^2.

4 Discussion

We have shown that the Berenstein-Vázquez Hamiltonian (11) is contained in the commuting transfer matrix (32), which directly implies the integrability of this Hamiltonian. One key ingredient is the left K-matrix (20), which is of the projected type [30]. Another key ingredient is the right K-matrix (25), obtained through a seldom-used isomorphism noted in [10]. We have also found expressions (43), (44), (54), (55) for the eigenvalues of this transfer matrix in terms of roots of the Bethe equations (56)-(58). We have therefore completed the generalization of important closed-chain results of Minahan and Zarembo [2] to the $Y=0$ open chain.

[7] Generally, there are fewer Bethe roots of types 2 and 3 for cases II and III in comparison with case I.

We have seen that the Bethe equations based on the vacuum eigenvalue (41) are not uniquely fixed – a certain "gauge" freedom exists. A similar phenomenon may occur for other integrable spin chains with symmetry group of rank greater than one. (This phenomenon is distinct from the well-known duality transformations of supersymmetric spin chains, see e.g. [31].) For case I in (55), the one-loop Bethe equations (56)-(58) agree with those obtained in [25] from the all-loop Bethe equations [16] by performing the weak-coupling limit and then reducing to the $SO(6)$ sector. Hence, our results provide support for those all-loop Bethe equations.

The analytical Bethe ansatz approach that we followed here is heuristic, and requires making some assumptions, in particular (54). It would be useful to carry out a more rigorous analysis based on nested algebraic Bethe ansatz. For the $O(2N)$ closed spin chain, a nested algebraic Bethe ansatz was developed in [32]. One would need to generalize that approach to the open chain with projected K-matrices considered here. Besides the difficulty of working out the necessary commutation relations, one can foresee the following further difficulty: after the first level of nesting, the "bulk" quantum space will have 4 dimensions (two less than the original 6 dimensions), while the "boundary" quantum spaces will still have 5 dimensions. That is, the bulk and boundary quantum spaces will not have the same dimension. The reduced transfer matrix will therefore not be given by the tensor product of two usual transfer matrices, as happens at the final stage in the $GL(N)$ case [26].

We have restricted our attention here to the $Y = 0$ maximal giant graviton brane, which is the simplest known integrable open string/chain example. It may be interesting to consider the corresponding problem in more complicated examples, such as the $Z = 0$ maximal giant graviton brane [7].

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A "Gauge" transformations among the cases

Let us denote by $c_i^{\rm I}(u), c_i^{\rm IIa}(u)$, etc. the functions (54) for cases I, IIa, etc. in (55), respectively. It is easy to see that the functions for case IIa are related to the corresponding ones for case

I as follows

$$
\begin{aligned}
c_1^{\mathrm{IIa}}(u) &= \left(\frac{u}{u-1}\right)^2 c_1^{\mathrm{I}}(u)\,, \\
c_2^{\mathrm{IIa}}(u) &= \left(\frac{u-2}{u-1}\right)^2 c_2^{\mathrm{I}}(u)\,, \\
c_3^{\mathrm{IIa}}(u) &= \left(\frac{u-2}{u-1}\right)^2 c_3^{\mathrm{I}}(u)\,, \\
c_4^{\mathrm{IIa}}(u) &= \left(\frac{u}{u-1}\right)^2 c_4^{\mathrm{I}}(u)\,.
\end{aligned}
\tag{60}
$$

Similarly, the functions for case IIb are related to the corresponding ones for case I as follows

$$
\begin{aligned}
c_1^{\mathrm{IIb}}(u) &= \left(\frac{u}{u-1}\right)^2 c_1^{\mathrm{I}}(u)\,, \\
c_2^{\mathrm{IIb}}(u) &= \left(\frac{u-2}{u-1}\right)^2 c_2^{\mathrm{I}}(u)\,, \\
c_3^{\mathrm{IIb}}(u) &= \left(\frac{u}{u-1}\right)^2 c_3^{\mathrm{I}}(u)\,, \\
c_4^{\mathrm{IIb}}(u) &= \left(\frac{u-2}{u-1}\right)^2 c_4^{\mathrm{I}}(u)\,.
\end{aligned}
\tag{61}
$$

Finally, the functions for case III are related to the corresponding ones for case I as follows

$$
\begin{aligned}
c_1^{\mathrm{III}}(u) &= \left(\frac{u}{u-1}\right)^4 c_1^{\mathrm{I}}(u)\,, \\
c_2^{\mathrm{III}}(u) &= \left(\frac{u-2}{u-1}\right)^4 c_2^{\mathrm{I}}(u)\,, \\
c_3^{\mathrm{III}}(u) &= \frac{u^2(u-2)^2}{(u-1)^4}\, c_3^{\mathrm{I}}(u)\,, \\
c_4^{\mathrm{III}}(u) &= \frac{u^2(u-2)^2}{(u-1)^4}\, c_4^{\mathrm{I}}(u)\,.
\end{aligned}
\tag{62}
$$

Let us now consider the expression (43) for $\Lambda(u)$, and identify the functions $c_i(u)$ there with $c_i^{\mathrm{I}}(u)$. Our observation is that, by making in that expression the simple transformation

$$
Q_3(u) \to u^2 Q_3(u)\,,
\tag{63}
$$

we obtain, in view of (60), the corresponding expression for $\Lambda(u)$ in terms of $c_i^{\mathrm{IIa}}(u)$. Similarly, by making instead the transformation

$$
Q_2(u) \to u^2 Q_2(u)\,,
\tag{64}
$$

we obtain, in view of (61), the expression for $\Lambda(u)$ in terms of $c_i^{\mathrm{IIb}}(u)$. Finally, in view of (62), by making both transformations (63) and (64), we obtain the expression for $\Lambda(u)$ in terms of $c_i^{\mathrm{III}}(u)$.

Similarly, let us consider the Bethe equations in their original form (49), (51), (53), and identify the functions $c_i(u)$ there with $c_i^{\mathrm{I}}(u)$. By making the transformations (63) and/or (64), we obtain the Bethe equations for the other cases.

In short, (63) and (64) are sorts of (discrete) "gauge" transformations that relate the four cases (55) of Bethe ansatz solutions. In our numerical investigations, we have observed that for case I there are generally more zero Bethe roots of types 2 and 3 in comparison with cases II and III, which is consistent with the presence of additional factors of u^2 in the corresponding Q functions.

Related transformations were briefly discussed in [26]. There, the transformations involve also $Q_1(u)$, and therefore, relate Bethe ansatz solutions based on reference states with different energies.

deg	$\{u_{1,k}\}$	$\{u_{2,k}\}$	$\{u_{3,k}\}$
9	–	–	–
6	$1/2$	0	–
4	$\sqrt{3}/6$	0	0
4	$\sqrt{3}/2$	0	0
2	$\pm i/2$	0	0

Table 1: Degeneracy and Bethe roots for case I with $L = 0$.

deg	$\{u_{1,k}\}$	$\{u_{2,k}\}$	$\{u_{3,k}\}$
16	–	–	–
16	$\sqrt{3}/6$	0	–
16	$\sqrt{3}/2$	0	–
9	$1/2$	–	–
9	$1/2$	0	0
9	$(\sqrt{2}+1)/2$	0	0
9	$(\sqrt{2}-1)/2$	0	0
6	$\pm i/2$	0	–
10	$\pm i/2$	0	0
14	0.230955, 0.668326	0	0
14	$0.716015 \pm 0.512521i$	0	0
14	$\sqrt{3}/6, \sqrt{3}/2$	$\sqrt{6}/3$	0
4	0.415511, 1.15211	1	1
1	$1/2, \pm i/2$	$0, i$	$0, i$
1	$1/2,\ 0.479716 \pm 0.971633i$	$0, 1.38848i$	$0, 1.38848i$
1	1/2, 0.208963, 1.02227	0, 0.767271	0, 0.767271
1	$1/2,\ 0.414496 \pm 0.502211i$	$0, 0.812907i$	$0, 0.812907i$

Table 2: Degeneracy and Bethe roots for case I with $L = 1$.

References

[1] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, *et al.*, "Review of AdS/CFT Integrability: An Overview," `arXiv:1012.3982 [hep-th]`. * Temporary entry *.

[2] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for $\mathcal{N} = 4$ super Yang-Mills," *JHEP* **03** (2003) 013, `arXiv:hep-th/0212208`.

[3] C. Sieg, "Review of AdS/CFT Integrability, Chapter I.2: The spectrum from perturbative gauge theory," `arXiv:1012.3984 [hep-th]`. * Temporary entry *.

[4] R. A. Janik, "Review of AdS/CFT Integrability, Chapter III.5: Lúscher corrections," `arXiv:1012.3994 [hep-th]`. * Temporary entry *.

[5] Z. Bajnok, "Review of AdS/CFT Integrability, Chapter III.6: Thermodynamic Bethe Ansatz," `arXiv:1012.3995 [hep-th]`. * Temporary entry *.

[6] K. Zoubos, "Review of AdS/CFT Integrability, Chapter IV.2: Deformations, Orbifolds and Open Boundaries," `arXiv:1012.3998 [hep-th]`. * Temporary entry *.

[7] D. M. Hofman and J. M. Maldacena, "Reflecting magnons," *JHEP* **11** (2007) 063, `arXiv:0708.2272 [hep-th]`.

[8] D. Berenstein and S. E. Vazquez, "Integrable open spin chains from giant gravitons," *JHEP* **06** (2005) 059, `arXiv:hep-th/0501078`.

[9] I. Cherednik, "Factorizing Particles on a Half Line and Root Systems," *Theor.Math.Phys.* **61** (1984) 977–983.

[10] E. Sklyanin, "Boundary Conditions for Integrable Quantum Systems," *J.Phys.A* **A21** (1988) 2375.

[11] S. Ghoshal and A. B. Zamolodchikov, "Boundary S matrix and boundary state in two-dimensional integrable quantum field theory," *Int.J.Mod.Phys.* **A9** (1994) 3841–3886, `arXiv:hep-th/9306002 [hep-th]`.

[12] A. Agarwal, "Open spin chains in super Yang-Mills at higher loops: Some potential problems with integrability," *JHEP* **08** (2006) 027, `arXiv:hep-th/0603067`.

[13] K. Okamura and K. Yoshida, "Higher loop Bethe ansatz for open spin-chains in AdS/CFT," *JHEP* **09** (2006) 081, `arXiv:hep-th/0604100`.

[14] C. Ahn and R. I. Nepomechie, "The Zamolodchikov-Faddeev algebra for open strings attached to giant gravitons," *JHEP* **05** (2008) 059, `arXiv:0804.4036 [hep-th]`.

[15] H.-Y. Chen and D. H. Correa, "Comments on the Boundary Scattering Phase," *JHEP* **02** (2008) 028, `arXiv:0712.1361 [hep-th]`.

[16] W. Galleas, "The Bethe Ansatz Equations for Reflecting Magnons," *Nucl. Phys.* **B820** (2009) 664–681, `arXiv:0902.1681 [hep-th]`.

[17] N. Beisert, "The S-Matrix of AdS/CFT and Yangian Symmetry," *PoS* **SOLVAY** (2006) 002, `arXiv:0704.0400 [nlin.SI]`.

[18] C. Ahn and R. I. Nepomechie, "Yangian symmetry and bound states in AdS/CFT boundary scattering," *JHEP* **1005** (2010) 016, `arXiv:1003.3361 [hep-th]`.

[19] N. MacKay and V. Regelskis, "Yangian symmetry of the Y=0 maximal giant graviton," *JHEP* **1012** (2010) 076, `arXiv:1010.3761 [hep-th]`.

[20] L. Palla, "Yangian symmetry of boundary scattering in AdS/CFT and the explicit form of bound state reflection matrices," *JHEP* **1103** (2011) 110, `arXiv:1102.0122 [hep-th]`.

[21] D. H. Correa and C. A. S. Young, "Finite size corrections for open strings/open chains in planar AdS/CFT," `arXiv:0905.1700 [hep-th]`.

[22] Z. Bajnok and L. Palla, "Boundary finite size corrections for multiparticle states and planar AdS/CFT," *JHEP* **01** (2011) 011, `arXiv:1010.5617 [hep-th]`.

[23] N. Mann and S. E. Vazquez, "Classical open string integrability," *JHEP* **04** (2007) 065, `arXiv:hep-th/0612038`.

[24] A. Dekel and Y. Oz, "Integrability of Green-Schwarz Sigma Models with Boundaries," *JHEP* **08** (2011) 004, `arXiv:1106.3446 [hep-th]`.

[25] R. I. Nepomechie, "Bethe ansatz equations for open spin chains from giant gravitons," *JHEP* **05** (2009) 100, `arXiv:0903.1646 [hep-th]`.

[26] R. I. Nepomechie, "Nested algebraic Bethe ansatz for open GL(N) spin chains with projected K-matrices," *Nucl.Phys.* **B831** (2010) 429–451, `arXiv:0911.5494 [hep-th]`.

[27] A. B. Zamolodchikov and A. B. Zamolodchikov, "Relativistic Factorized S Matrix in Two-Dimensions Having O(N) Isotopic Symmetry," *Nucl.Phys.* **B133** (1978) 525.

[28] A. Lima-Santos and R. Malara, "C**(1)(n), D**(1)(n) and A**(2)(2n-1) reflection K-matrices," *Nucl.Phys.* **B675** (2003) 661–684, `arXiv:nlin/0307046 [nlin-si]`.

[29] E. Abdalla and A. Lima-Santos, "Integrable models: from dynamical solutions to string theory," `arXiv:0808.3919 [hep-th]`.

[30] H. Frahm and N. A. Slavnov, "New solutions to the reflection equation and the projecting method," *J.Phys.A* **A32** (1999) 1547–1556, `arXiv:cond-mat/9810312 [cond-mat]`.

[31] N. Beisert, V. A. Kazakov, K. Sakai, and K. Zarembo, "Complete spectrum of long operators in $\mathcal{N} = 4$ SYM at one loop," *JHEP* **07** (2005) 030, `arXiv:hep-th/0503200`.

[32] M. Martins and P. Ramos, "The Algebraic Bethe ansatz for rational braid - monoid lattice models," *Nucl.Phys.* **B500** (1997) 579–620, `arXiv:hep-th/9703023 [hep-th]`.